香　港　中　文　大　學
The Chinese University of Hong Kong

IMSC 2048 — Practice Final Examination

Time allowed: 3 hours

This practice exam has **8 questions** totalling **100 marks**. Questions 1–2 are on bilinear forms, Questions 3–4 are on group representations, and Questions 5–8 are on field extensions and Galois theory.

Question	**Topic**	**Marks**
1	Bilinear Forms	10
2	Bilinear Forms	10
3	Group Representations	15
4	Group Representations	15
5	Field Extensions	10
6	Galois Theory	10
7	Galois Theory	15
8	Galois Theory	15
Total		**100**

Question 1 (10 marks)

Let $V = \mathbb{R}^3$ and let $\beta\colon V \times V \to \mathbb{R}$ be the symmetric bilinear form whose Gram matrix with respect to the standard basis $\{e_1, e_2, e_3\}$ is

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(a) Prove that β is positive definite.

(b) Apply the Gram–Schmidt process to the standard basis $\{e_1, e_2, e_3\}$ with respect to β to obtain an orthonormal basis $\{f_1, f_2, f_3\}$ of V satisfying $\beta(f_i, f_j) = \delta_{ij}$.

Question 2 (10 marks)

Let $V = \mathbb{R}^4$ and let $\beta\colon V \times V \to \mathbb{R}$ be the symmetric bilinear form whose Gram matrix with respect to the standard basis is

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

(a) Is β nondegenerate? Justify your answer.

(b) Determine the signature (p, q) of β, where p is the positive index of inertia and q is the negative index of inertia.

(c) Find the maximal dimension of a subspace $W \subseteq V$ such that $\beta(w, w) = 0$ for all $w \in W$.

Question 3 (15 marks)

Let $D_4 = \langle r, s \mid r^4 = s^2 = e,\ srs^{-1} = r^{-1} \rangle$ be the dihedral group of order 8.

(a) List all conjugacy classes of D_4 and their sizes.

(b) Write down the complete character table of D_4 and verify the orthogonality relations for two distinct rows.

(c) Consider the subgroup $H = \langle r \rangle \cong C_4$. Write down the character table of C_4 and decompose the restrictions of each irreducible representation of D_4 to H into irreducible representations of C_4.

Question 4 (15 marks)

Let G be a finite group and let χ be an irreducible character of G. Define the *centre* of χ by

$$Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(e)\}.$$

(a) If G is abelian group, show that $Z(\chi) = G$.

(b) For general finite group G (not necessarily abelian), prove that

$$\chi(e)^2 \leq [G : Z(\chi)]$$

where $[G : Z(\chi)] = |G|/|Z(\chi)|$.

(c) Show that equality $\chi(e)^2 = [G : Z(\chi)]$ holds if and only if $\chi(g) = 0$ for all $g \in G \setminus Z(\chi)$.

Question 5 (10 marks)
For each of the following field extensions, find the degree $[K : F]$, and determine whether the extension is separable or inseparable, whether is normal or not, and whether it is Galois or not. Justify your answers.

(a) $F = \mathbb{F}_3(t), \quad K = F[x]/(x^3 - t)$.

(b) $F = \mathbb{Q}, \quad K = \mathbb{Q}(\sqrt[4]{2})$.

(c) Let $F = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ denote the field with p elements, where p is a prime integer. Let $K = F[x]/(x^p - x + 1)$.

Question 6 (10 marks)

Let $\zeta = e^{2\pi i/8}$ be a primitive 8th root of unity and let $K = \mathbb{Q}(\zeta)$.

(a) Find the minimal polynomial of ζ over $\mathbb{Q}$ and determine $[K : \mathbb{Q}]$.

(b) Show that $K = \mathbb{Q}(\zeta)$ is the splitting field of $x^8 - 1$ over $\mathbb{Q}$, and conclude that $K/\mathbb{Q}$ is a Galois extension.

(c) Describe the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ by listing each automorphism $\sigma_a \colon \zeta \mapsto \zeta^a$ explicitly. Identify G as a familiar group in terms of direct product of cyclic groups.

Question 7 (15 marks)

Let $K/\mathbb{Q}$ be a Galois extension over field of rational numbers. Assume that $G = \mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(a) Show that $K = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ for some $a, b \in \mathbb{Q}$ such that $\sqrt{a} \notin \mathbb{Q}[\sqrt{b}]$ and $\sqrt{b} \notin \sqrt{\Game}$.

(b) Show how the Galois group G acts on $\sqrt{a}$ and $\sqrt{b}$.

(c) When $a = 2$ and $b = 3$, find all intermediate fields between $\mathbb{Q}$ and K (including $\mathbb{Q}$ and K itself) by writing them as $\mathbb{Q}(\alpha)$ for some explicit $\alpha \in K$. (You also need to write down K in this form as well.)

Question 8 (15 marks)

Consider the polynomial $f(x) = x^4 - 2 \in \mathbb{Q}[x]$.

(a) Show that f is irreducible over $\mathbb{Q}$.

(b) Let K be the splitting field of f over $\mathbb{Q}$. Show that $K = \mathbb{Q}(\sqrt[4]{2}, i)$ and find $[K : \mathbb{Q}]$.

(c) Determine $G = \mathrm{Gal}(K/\mathbb{Q})$ by writing down generators. Show that $G \cong D_4$, the dihedral group of order 8. (*Hint*: consider the automorphisms $\sigma\colon \sqrt[4]{2} \mapsto i\sqrt[4]{2},\ i \mapsto i$ and $\tau\colon \sqrt[4]{2} \mapsto \sqrt[4]{2},\ i \mapsto -i$.)

(d) Is D_4 solvable? Determine whether $f(x) = 0$ is solvable by radicals over $\mathbb{Q}$, and justify your answer.

Write your answers here

— End of Practice Final Examination —